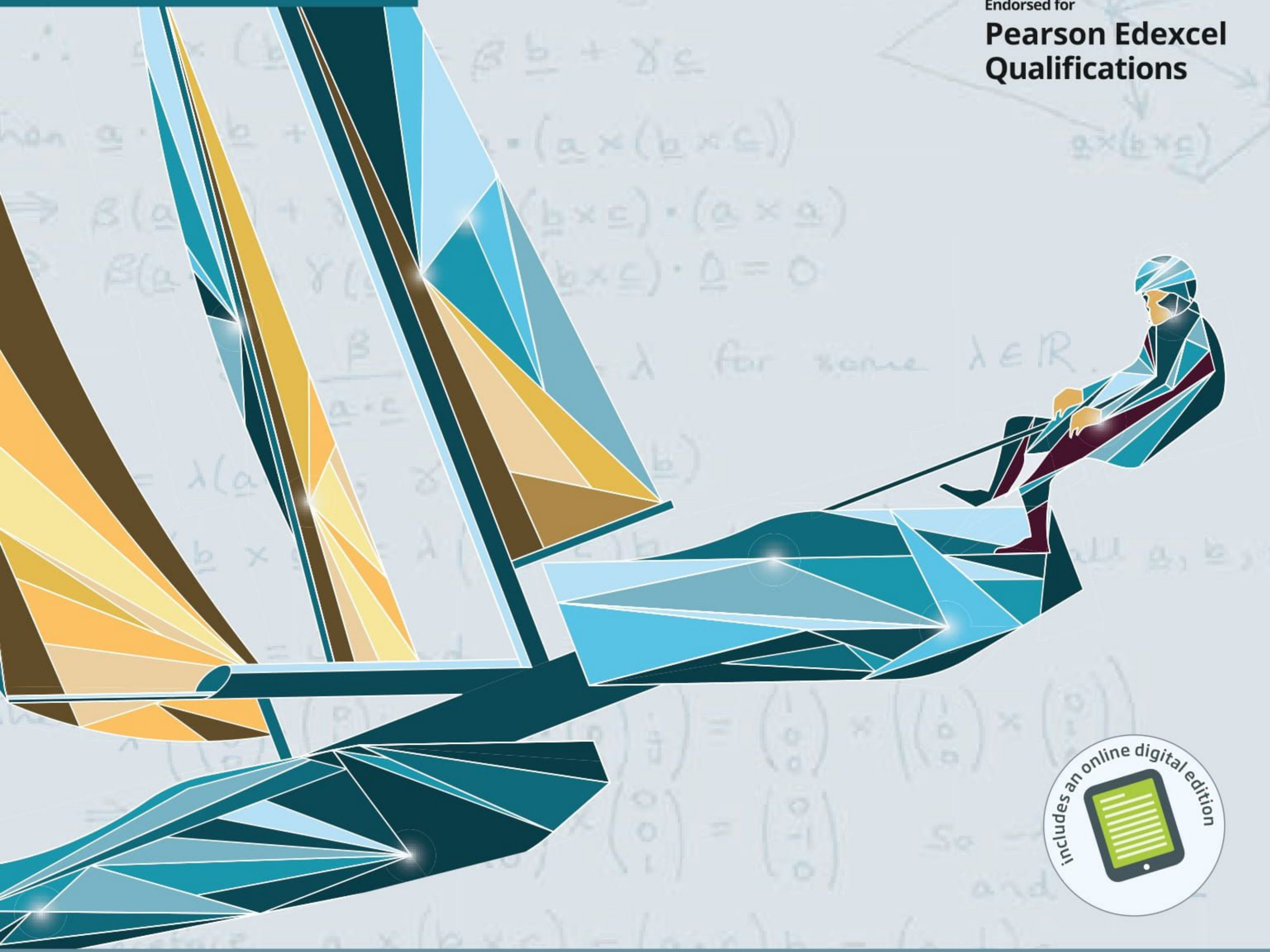
11 - 19 PROGRESSION

Endorsed for
Pearson Edexcel Qualifications



Pearson Edexcel AS and A level Further Mathematics

Further Pure Mathematics 1

FP1

Pearson

Contents

● = A level only

Overarching themes

The following three overarching themes have been fully integrated throughout the Pearson Edexcel AS and A level Mathematics series, so they can be applied alongside your learning and practice.

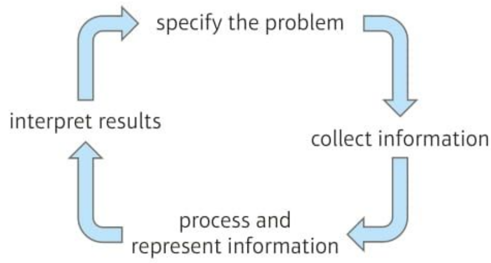
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols
- Dedicated sections on mathematical proof explain key principles and strategies
- Opportunities to critique arguments and justify methods

2. Mathematical problem solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Structured and unstructured questions to build confidence
- Challenge boxes provide extra stretch

The Mathematical Problem-solving cycle



3. Mathematical modelling

- Dedicated modelling sections in relevant topics provide plenty of practice where you need it
- Examples and exercises include qualitative questions that allow you to interpret answers in the context of the model
- Dedicated chapter in Statistics & Mechanics Year 1/AS explains the principles of modelling in mechanics

Finding your way around the book

Access an online digital edition using the code at the front of the book.

Each chapter starts with a list of objectives

The real world applications of the maths you are about to learn are highlighted at the start of the chapter with links to relevant questions in the chapter

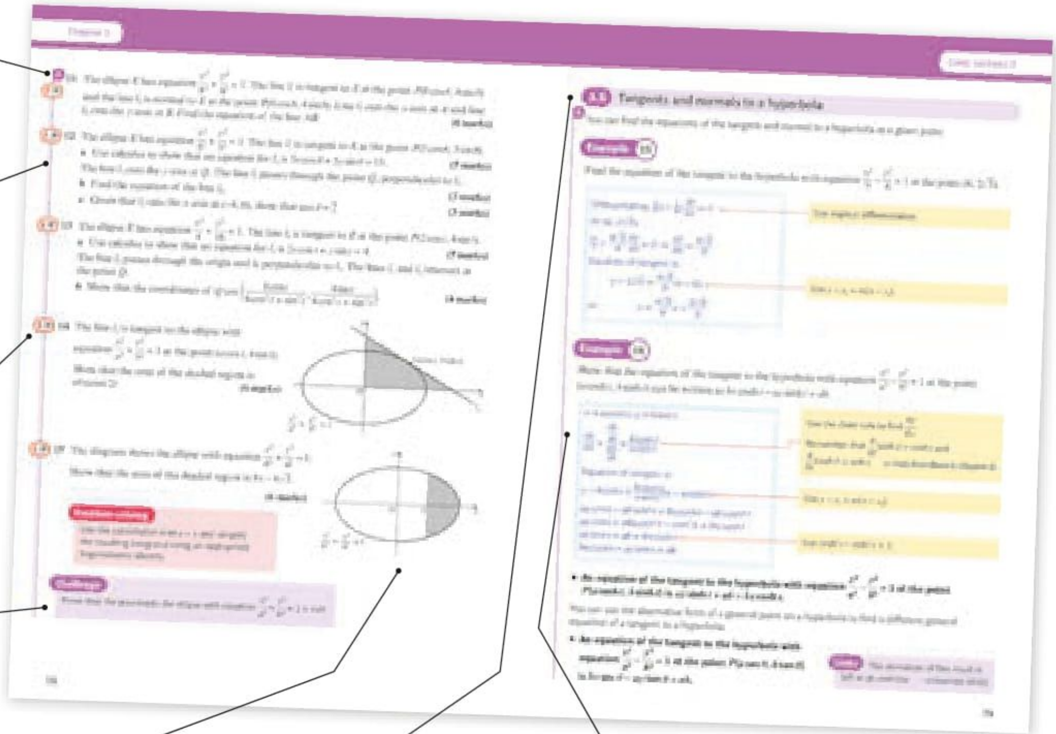
The *Prior knowledge check* helps make sure you are ready to start the chapter

A level content is clearly flagged

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Challenge boxes give you a chance to tackle some more difficult questions



Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

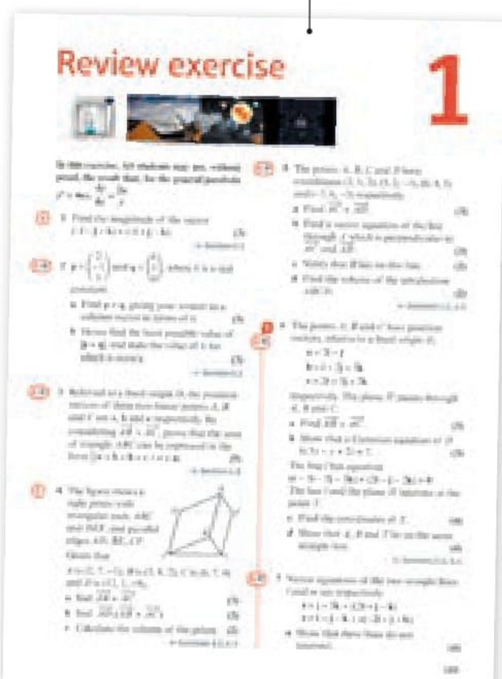
Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

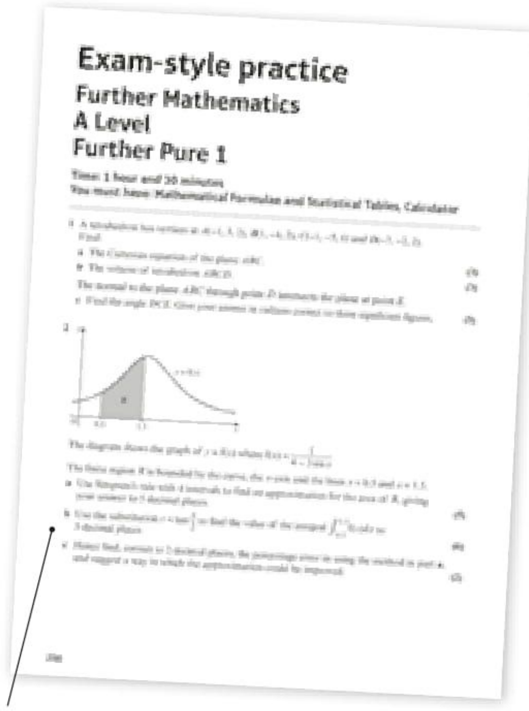
Each section begins with explanation and key learning points

Step-by-step worked examples focus on the key types of questions you'll need to tackle

Each chapter ends with a *Mixed exercise* and a *Summary of key points*

Every few chapters a *Review exercise* helps you consolidate your learning with lots of exam-style questions





AS and A level practice papers at the back of the book help you prepare for the real thing.

Extra online content

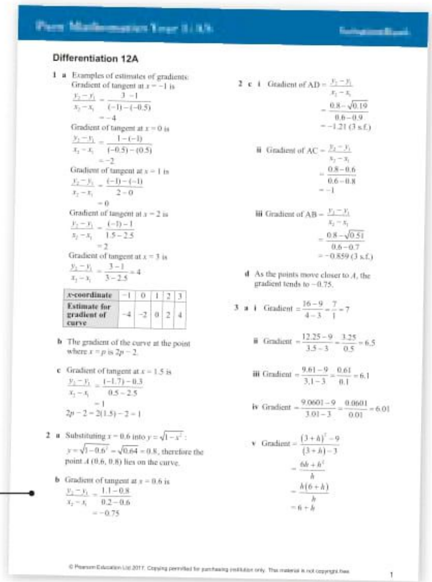
Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides a full worked solution for every question in the book.

Online Full worked solutions are available in SolutionBank.

Download all the solutions as a PDF or quickly find the solution you need online


Differentiation 12A

Use of technology

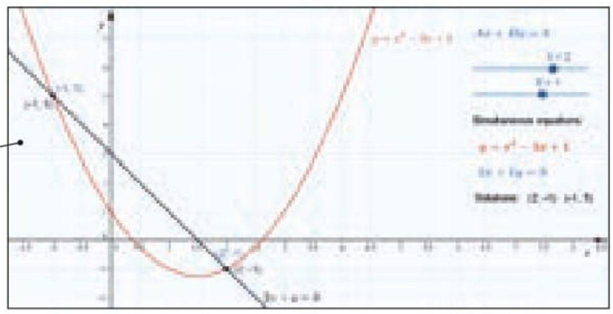
Explore topics in more detail, visualise problems and consolidate your understanding using pre-made GeoGebra activities.

Online Find the point of intersection graphically using technology.

GeoGebra

GeoGebra-powered interactives

Interact with the maths you are learning using GeoGebra's easy-to-use tools



Access all the extra online content for free at:

www.pearsonschools.co.uk/fp1maths

You can also access the extra online content by scanning this QR code:

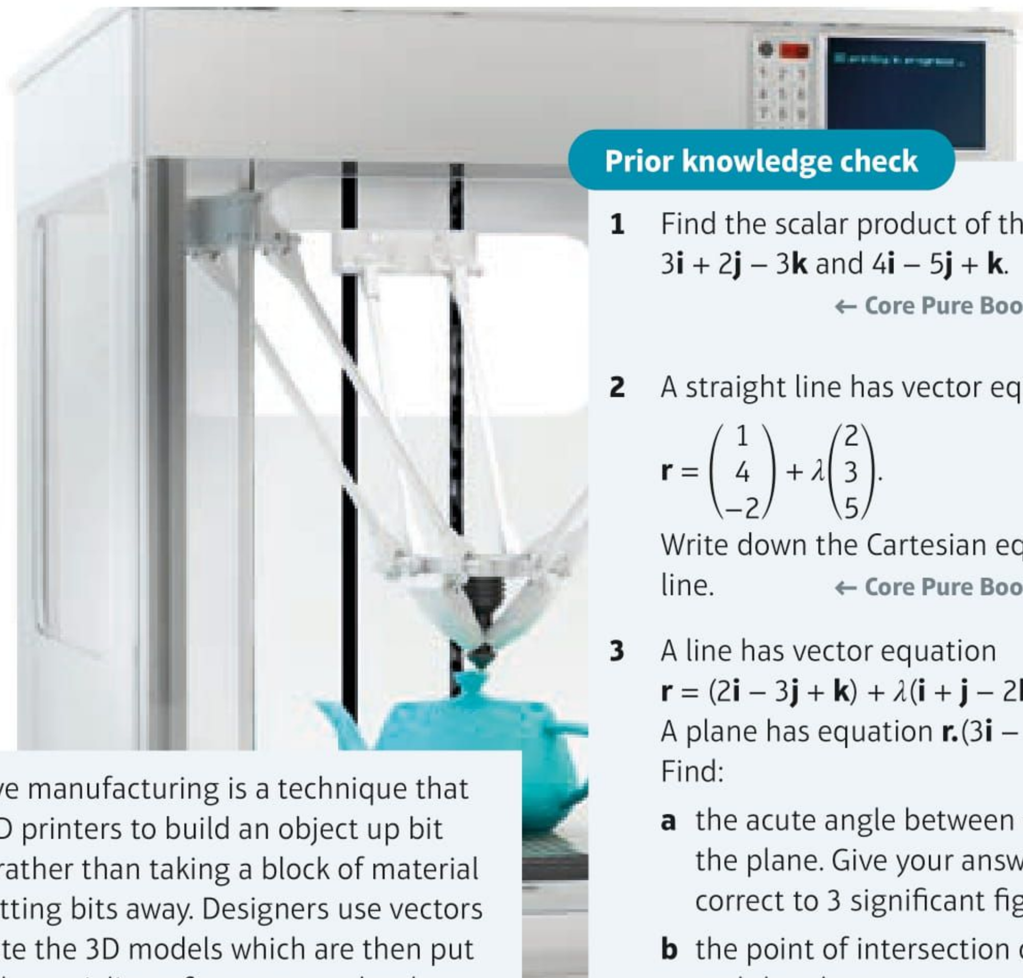
Vectors

1

Objectives

After completing this chapter you should be able to:

- Find the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ → pages 2–6
- Interpret $|\mathbf{a} \times \mathbf{b}|$ as an area → pages 7–11
- Find the scalar triple product $\mathbf{a}.\mathbf{b} \times \mathbf{c}$ of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and be able to interpret it as a volume → pages 11–16
- Write the vector equation of a line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ → pages 16–20
- Find the direction ratios and direction cosines of a line → pages 17–20
- Use vectors in problems involving points, lines and planes and use the equivalent Cartesian forms for the equations of lines and planes → pages 20–25



Prior knowledge check

1 Find the scalar product of the vectors $3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$. ← Core Pure Book 1, Section 9.3

2 A straight line has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

Write down the Cartesian equation of the line. ← Core Pure Book 1, Section 9.1

3 A line has vector equation $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.
A plane has equation $\mathbf{r}.(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 2$.
Find:

a the acute angle between the line and the plane. Give your answer in radians correct to 3 significant figures.

b the point of intersection of the line and the plane. ← Core Pure Book 1, Sections 9.4, 9.5

Additive manufacturing is a technique that uses 3D printers to build an object up bit by bit rather than taking a block of material and cutting bits away. Designers use vectors to create the 3D models which are then put through specialist software to render the object printable. → Exercise 1C Q11

1.1 Vector product

You have already encountered the **scalar** (or **dot**) **product** of two vectors.

The scalar (or dot) product of two vectors **a** and **b** is written as **a.b**, and defined as

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta,$$

where θ is the angle between **a** and **b**.

The scalar product produces a number (or scalar) as an answer. It is useful to define a second type of product that gives an answer as a vector.

Links If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ then $\mathbf{a}.\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$.
← Core Pure Book 1, Chapter 9

Online Use GeoGebra to explore the cross product of two vectors.

- **The vector (or cross) product of the vectors a and b is defined as**

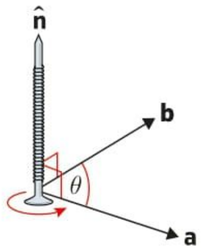
  $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

  where θ is the angle between a and b.

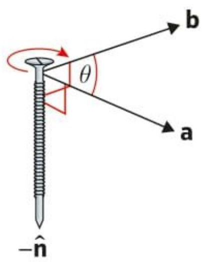
Notation $\hat{\mathbf{n}}$ is the unit vector that is perpendicular to both **a** and **b**.

Since $0 \leqslant \theta \leqslant 180°$, $|\mathbf{a}||\mathbf{b}|\sin\theta$ is a positive scalar quantity. This means that $\mathbf{a} \times \mathbf{b}$ is a vector quantity with magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$ that acts in the direction of $\hat{\mathbf{n}}$.

The direction of $\hat{\mathbf{n}}$ is that in which a right-handed screw would move when turned from **a** to **b**.



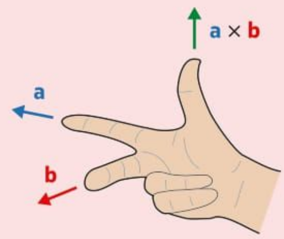
If the turn is in the opposite sense, i.e. from **b** to **a**, then the movement of the screw is in the opposite direction to $\hat{\mathbf{n}}$, i.e. in the direction of $-\hat{\mathbf{n}}$.



$$\begin{aligned} \text{So } \mathbf{b} \times \mathbf{a} &= |\mathbf{b}||\mathbf{a}|\sin\theta\,(-\hat{\mathbf{n}}) \\ &= -|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} \\ &= -\mathbf{a} \times \mathbf{b} \end{aligned}$$

- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$

Problem-solving You can also use a 'right-hand rule' to determine the direction of $\hat{\mathbf{n}}$, and hence the direction of $\mathbf{a} \times \mathbf{b}$.
If **a** is your first finger, and **b** is your second finger, then $\mathbf{a} \times \mathbf{b}$ acts in the direction of your thumb:



Watch out The vector product is not commutative: the order of multiplication matters.

Example 1

Find the values of:

a $\mathbf{i} \times \mathbf{i}$ **b** $\mathbf{j} \times \mathbf{k}$ **c** $\mathbf{i} \times \mathbf{k}$.

a $\mathbf{i} \times \mathbf{i} = \mathbf{0}$

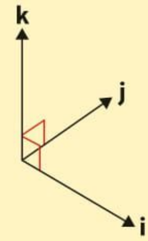
sin $\theta = 0$, as the angle between **i** and itself is zero.

b $\mathbf{j} \times \mathbf{k} = 1 \times 1 \times \sin 90° \mathbf{i} = \mathbf{i}$

The angle between **j** and **k** is 90° and, as **j** and **k** are unit vectors, each has magnitude 1 unit.

c $\mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -1 \times 1 \times \sin 90° \mathbf{j} = -\mathbf{j}$

Use the right-hand rule. If **i** is your first finger and **k** is your second finger, your thumb will point **away** from **j**, so $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.



- $\mathbf{i} \times \mathbf{i} = \mathbf{0}$
- $\mathbf{j} \times \mathbf{j} = \mathbf{0}$
- $\mathbf{k} \times \mathbf{k} = \mathbf{0}$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ **and** $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ **and** $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ **and** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

As $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}}$, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ implies that $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\sin \theta = 0$.

$\sin \theta = 0$ implies that $\theta = 0$ or 180°, so **a** and **b** must be parallel.

- **If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then either $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or a and b are parallel.**

Example 2

Given that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ find $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) \\
&\quad + a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) \\
&\quad + a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k}) \\
&= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3(\mathbf{i}) + a_3b_1(\mathbf{j}) + a_3b_2(-\mathbf{i}) \\
&= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}
\end{aligned}$$

Simplify the cross product and collect like terms.

In determinant form,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

You can write each component as the determinant of a 2 × 2 matrix, or the whole vector product as a determinant of a 3 × 3 matrix.

← Core Pure Book 1, Chapter 6

Notation You may assume the vector product is **distributive** over vector addition. This means that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

- $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Example 3

Given that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$:

a directly

b by a method involving a determinant.

c Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.

a $(2\mathbf{i} - 3\mathbf{j}) \times (4\mathbf{i} + \mathbf{j} - \mathbf{k})$

$= 8(\mathbf{i} \times \mathbf{i}) + 2(\mathbf{i} \times \mathbf{j}) - 2(\mathbf{i} \times \mathbf{k}) - 12(\mathbf{j} \times \mathbf{i}) - 3(\mathbf{j} \times \mathbf{j}) + 3(\mathbf{j} \times \mathbf{k})$

$= \mathbf{0} + 2\mathbf{k} + 2\mathbf{j} + 12\mathbf{k} - \mathbf{0} + 3\mathbf{i}$

$= 3\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$

Use the distributive property to multiply out the brackets.

Simplify the cross products of unit vectors.

b $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 4 & 1 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & 0 \\ 1 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 0 \\ 4 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix}$

$= \mathbf{i}(3 - 0) - \mathbf{j}(-2 - 0) + \mathbf{k}(2 + 12)$

$= 3\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}$

Problem-solving

Using the discriminant is usually a quicker way to evaluate the cross product.

c $(3\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}).(2\mathbf{i} - 3\mathbf{j}) = (3 \times 2) + (2 \times (-3)) + (14 \times 0) = 0$

$(3\mathbf{i} + 2\mathbf{j} + 14\mathbf{k}).(4\mathbf{i} + \mathbf{j} - \mathbf{k}) = (3 \times 4) + (2 \times 1) + (14 \times (-1)) = 0$

Work out $(\mathbf{a} \times \mathbf{b}).\mathbf{a}$ and $(\mathbf{a} \times \mathbf{b}).\mathbf{b}$. If both answers are 0 then $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.

Example 4

Find a unit vector perpendicular to both $(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $(8\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$.

The vector product will give a perpendicular vector.

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 2 \\ 8 & 3 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 4 & 2 \\ 8 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 4 & 3 \\ 8 & 3 \end{vmatrix}$

$= \mathbf{i}(9 - 6) - \mathbf{j}(12 - 16) + \mathbf{k}(12 - 24)$

$= 3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

Since $|3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}| = \sqrt{3^2 + 4^2 + (-12)^2} = 13$

a suitable unit vector is $\frac{1}{13}(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k})$.

Watch out You can find vector products using your calculator. But you might encounter a vector with an unknown in it, so it is important that you know how to find the vector product manually.

Find the magnitude of your product vector.

Divide the vector by its magnitude to obtain a unit vector.

Example 5

Find the sine of the acute angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -3\mathbf{j} + 4\mathbf{k}$.

$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$

So $\dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \sin\theta$

Rearrange the formula to make $\sin\theta$ the subject. $|\hat{\mathbf{n}}| = 1$ so $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 0 & -3 & 4 \end{vmatrix}$$

$$= \mathbf{i}(4 + 6) - \mathbf{j}(8 - 0) + \mathbf{k}(-6 - 0)$$

$$= 10\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$$

Calculate the vector product.

and $|10\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}| = \sqrt{100 + 64 + 36}$

Find the magnitude of $\mathbf{a} \times \mathbf{b}$.

So $\sin\theta = \dfrac{\sqrt{200}}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{(-3)^2 + 4^2}}$

Also find the magnitude of **a** and of **b** and substitute the three surds into the formula for $\sin\theta$.

$$= \frac{\sqrt{200}}{\sqrt{9}\sqrt{25}}$$

$$= \frac{10\sqrt{2}}{3 \times 5}$$

$$= \frac{2\sqrt{2}}{3}$$

Simplify your answer.

Watch out In general, to find the angle between two vectors use the scalar product. This gives the cosine of the angle. Immediately we know whether the angle is acute or obtuse. In this example it is not clear whether the angle θ is acute or obtuse. This is similar to the ambiguous case when using the sine rule.

Exercise 1A

1 Simplify:

a $5\mathbf{j} \times \mathbf{k}$ **b** $3\mathbf{i} \times \mathbf{k}$ **c** $\mathbf{k} \times 3\mathbf{i}$

d $3\mathbf{i} \times (9\mathbf{i} - \mathbf{j} + \mathbf{k})$ **e** $2\mathbf{j} \times (3\mathbf{i} + \mathbf{j} - \mathbf{k})$ **f** $(3\mathbf{i} + \mathbf{j} - \mathbf{k}) \times 2\mathbf{j}$

g $\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ **h** $\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ **i** $\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ **j** $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

2 Find the vector product of the vectors **a** and **b**, leaving your answers in terms of λ in each case.

a $\mathbf{a} = \lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ $\mathbf{b} = \mathbf{i} - 3\mathbf{k}$

b $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ $\mathbf{b} = \mathbf{i} - \lambda\mathbf{j} + 3\mathbf{k}$

3 Find a unit vector that is perpendicular to both $2\mathbf{i} - \mathbf{j}$ and to $4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

4 Find a unit vector that is perpendicular to both $4\mathbf{i} + \mathbf{k}$ and $\mathbf{j} - \sqrt{2}\mathbf{k}$.

5 Find a unit vector that is perpendicular to both $\mathbf{i} - \mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$.

6 Find a unit vector that is perpendicular to both $\begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$ and to $\begin{pmatrix} 5 \\ 9 \\ 8 \end{pmatrix}$.

A E/P **17** A telephone wire is modelled as a straight line in 3D space. **i** and **j** are the horizontal vectors due east and north respectively, and **k** is the vertical unit vector. The units are metres.

An engineer inspects the wire at the point with position vector 6**k**, and finds that it is horizontal, and directed on a bearing of 015°.

a Find a vector equation of the wire, giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. **(4 marks)**

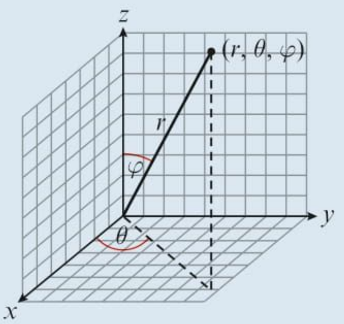
b Hence show that the wire will intersect with a second wire with vector equation

$$\left(\mathbf{r} - \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}\right) \times \begin{pmatrix} 5 - 2(\sqrt{6} - \sqrt{2}) \\ 2 - 2(\sqrt{6} + \sqrt{2}) \\ -5 \end{pmatrix} = \mathbf{0}$$

(3 marks)

c Give a possible criticism of this model. **(1 mark)**

Challenge



Spherical polar coordinates are defined by the distance from the origin, r, the 'azimuthal angle' (measured anti-clockwise from the x-axis in the xy-plane), θ, and the 'polar angle' (measured from the positive z-axis), φ.

A line L passes through the origin and the point with spherical polar coordinates $\left(3, \frac{\pi}{4}, \frac{\pi}{3}\right)$.

a Find, in their simplest form, the direction cosines of L.

b Find, in terms of θ and φ, expressions for the direction cosines of the line which passes through the origin and the point with spherical coordinates (r, θ, φ).

1.5 Solving geometrical problems

You can use the fact that the vector product **a** × **b** is perpendicular to both **a** and **b** to solve problems involving planes and lines in three dimensions.

Example 18

a Find, in the form **r.n** = p, an equation of the plane which contains the line l and the point with position vector **a** where l has equation $\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

b Give the equation of the plane in Cartesian form.

A

a The vector $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $\mathbf{n}$.

The vector $4\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$ also lies in the plane and is also perpendicular to $\mathbf{n}$, i.e. $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is perpendicular to $\mathbf{n}$.

So $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ 1 & -2 & 3 \end{vmatrix}$

$= 4\mathbf{i} + 2\mathbf{j}$

So the equation of the required plane is

$\mathbf{r}.(4\mathbf{i} + 2\mathbf{j}) = (4\mathbf{i} + 3\mathbf{j} + \mathbf{k}).(4\mathbf{i} + 2\mathbf{j})$

$\Rightarrow \mathbf{r}.(4\mathbf{i} + 2\mathbf{j}) = 16 + 6$

An equation of the plane is $\mathbf{r}.(4\mathbf{i} + 2\mathbf{j}) = 22$

b In Cartesian form this may be written as

$4x + 2y = 22$

$\Rightarrow 2x + y = 11$

Line l lies in the plane. The direction of l is $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and so this vector is perpendicular to $\mathbf{n}$.

The point (4, 3, 1) lies in the plane, and the point (3, 5, −2) lies on the line and so also in the plane, so the vector joining these two points also lies in the plane.

This vector $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is also perpendicular to $\mathbf{n}$.

$\mathbf{n}$ is in the direction of the vector product of $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Replace $\mathbf{r}$ with $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and perform the scalar product.

Example 19

Find a Cartesian equation of the plane that passes through the points $A(1, 0, -1)$, $B(2, 1, 0)$ and $C(2, 16, 6)$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{i} + 16\mathbf{j} + 7\mathbf{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 16 & 7 \end{vmatrix}$

$= -9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}$

So $\mathbf{r}.(-9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}) = (\mathbf{i} - \mathbf{k}).(-9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k})$

$\Rightarrow \mathbf{r}.(-9\mathbf{i} - 6\mathbf{j} + 15\mathbf{k}) = -9 - 15 = -24$

So the equation of the plane may be written as

$\mathbf{r}.(3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 8$

$\Rightarrow (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).(3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 8$

$\Rightarrow 3x + 2y - 5z = 8$, which is a Cartesian equation of the plane.

This is the direction of the normal to the plane.

Use $\mathbf{r.n} = \mathbf{a.n}$, where $\mathbf{a} = \mathbf{i} - \mathbf{k}$

Replace $\mathbf{r}$ by $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to obtain the Cartesian equation.

You may wish to check that each point lies on this plane.

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